Solution algorithm for pressure-velocity coupling in steady flow
-The segregate solver
(SIMPLE, SIMPLE+R, SIMPLE+C, PISO)

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Segregated solver

Direct-coupled solver is resource-demanding, become unaffordable for large cases, we prefer to decouple the pressure and velocity equations.
Different segregated solver

- SIMPLE
  - Semi-Implicit Method for Pressure-Linked Equations
- SIMPLER
  - Revisit
- SIMPLEC
  - Consistent
- PISO
  - Pressure Implicit with splitting of Operator
Part 1

• SIMPLE
  – Derivation of SIMPLE algorithm
    • Using summation form
    • Using matrix form
  – Derivation of the discretized pressure-correction equation in SIMPLE algorithm
    • Boundary issues

• Step by step demonstration of SIMPLE procedure on a 3x3 cells grid.
The SIMPLE algorithm at interior cells (allowing $b_o^p = 0$)

(Blue color means frozen or known value.)

Our goal

\[
\begin{align*}
    a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb} + \frac{p_e - p_o}{\Delta x} + b_o^u &= 0 \\
    a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb} + \frac{p_n - p_o}{\Delta y} + b_o^v &= 0 \\
    u_o - u_w + \frac{v_o - v_s}{\Delta x} &= 0 \\
    \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

Guessed, known value

approx.

\[
\begin{align*}
    a_o^u u_o^* + \sum_{nb} a_{nb}^u u_{nb}^* + \frac{p_e^* - p_o^*}{\Delta x} + b_o^u &= 0 \\
    a_o^v v_o^* + \sum_{nb} a_{nb}^v v_{nb}^* + \frac{p_n^* - p_o^*}{\Delta y} + b_o^v &= 0
\end{align*}
\]

Take the difference between ideal eq. and the approx. one

\[
\begin{align*}
    u &= u^* + u' \\
    v &= v^* + v' \\
    p &= p^* + p'
\end{align*}
\]

From here, we are going to derive an equation for pressure correction!

\[
\begin{align*}
    a_o^u (u_o - u_o^*) + \frac{p_e' - p_o'}{\Delta x} &= 0 \\
    a_o^v (v_o - v_o^*) + \frac{p_n' - p_o'}{\Delta y} &= 0 \\
    u_o - u_w + \frac{v_o - v_s}{\Delta x} &= 0 \\
    \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

\[
\begin{align*}
    a_o^u u_o' + \sum_{nb} a_{nb}^u u_{nb}' + \frac{p_e' - p_o'}{\Delta x} &= 0 \\
    a_o^v v_o' + \sum_{nb} a_{nb}^v v_{nb}' + \frac{p_n' - p_o'}{\Delta y} &= 0
\end{align*}
\]

\[u',v',p'\] are still coupled, difficult to handle, then neglect neighboring terms (main assumption)
Viewing A and B

\[ A^\text{diag}_{N_{uv} \times N_{uv}} + A^\text{nb}_{N_{uv} \times N_{uv}} \]

\[ A_{N_{uv} \times N_p} \]

\[ A_{N_p \times N_{uv}} \]

\[ B_{N_{uv}} \]

\[ B_{N_p} \]

nz = 3

nz = 87
Our goal

\[
\begin{bmatrix}
A_{Nuv \times Nuv}^{\text{diag}} + A_{Nuv \times Nuv}^{\text{nb}} \\
A_{Np \times Nuv}
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
+ A_{Nuv \times Np} \begin{bmatrix} p \end{bmatrix} + B_{Nuv} = 0
\]

approx.

\[
A_{Np \times Nuv} \begin{bmatrix} u \\
v
\end{bmatrix} + B_{Np} = 0
\]

The difference

\[
\begin{bmatrix}
u \\
v
\end{bmatrix} = \begin{bmatrix} u^* \\
v^*
\end{bmatrix} + \begin{bmatrix} u' \\
v'
\end{bmatrix}
\]

\[
\begin{bmatrix}
u \\
v
\end{bmatrix} = \begin{bmatrix} u \\
v
\end{bmatrix} - \begin{bmatrix} u^* \\
v^*
\end{bmatrix} + \begin{bmatrix} u' \\
v'
\end{bmatrix}
\]

\[
\begin{bmatrix}
u \\
v
\end{bmatrix} = \begin{bmatrix} u \\
v
\end{bmatrix} - \begin{bmatrix} u^* \\
v^*
\end{bmatrix} + \begin{bmatrix} u' \\
v'
\end{bmatrix}
\]

\[
\begin{bmatrix}
u \\
v
\end{bmatrix} = \begin{bmatrix} u \\
v
\end{bmatrix} - \begin{bmatrix} u^* \\
v^*
\end{bmatrix} + \begin{bmatrix} u' \\
v'
\end{bmatrix} = 0
\]

\[
A_{Nuv \times Nuv} \begin{bmatrix} u \\
v
\end{bmatrix} = -B_{Np}
\]

\[
A_{Nuv \times Nuv} \begin{bmatrix} u \\
v
\end{bmatrix} = -B_{Np}
\]

From here, we are going to derive an equation for pressure correction!
Derive the pressure-correction eq. in SIMPLE algorithm
(for interior cell, blue color means frozen or known value.)

\[
\begin{align*}
    a_o^u (u_o - u_o^*) + \frac{p_e' - p_o'}{\Delta x} &= 0 \\
    a_o^v (v_o - v_o^*) + \frac{p_n' - p_o'}{\Delta y} &= 0 \\
    \frac{u_o - u_w}{\Delta x} + \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

\[
\begin{align*}
    u_o &= u_o^* + \frac{p_e' - p_o'}{-a_o^u(u_o)\Delta x} \\
    v_o &= v_o^* + \frac{p_n' - p_o'}{-a_o^v(v_o)\Delta y}
\end{align*}
\]

Shift index

\[
\begin{align*}
    u_w &= u_w^* + \frac{p_o' - p_w'}{-a_o^u(u_w)\Delta x} \\
    v_s &= v_s^* + \frac{p_o' - p_s'}{-a_o^v(v_s)\Delta y}
\end{align*}
\]

Pressure correction eq. for **interior cells at some distant away from boundary cells**

\[
b^{p*'} = \frac{u_o^* - u_w^*}{\Delta x} + \frac{v_o^* - v_s^*}{\Delta y}
\]

\[
\begin{align*}
    \frac{p_e' - p_o'}{-a_o^u(u_o)\Delta x^2} - \frac{p_o' - p_w'}{-a_o^u(u_w)\Delta x^2} + \frac{p_n' - p_o'}{-a_o^v(v_o)\Delta y^2} - \frac{p_o' - p_s'}{-a_o^v(v_s)\Delta y^2} &= -b^{p*'}
\end{align*}
\]
The pressure-correction eq. in SIMPLE algorithm
(at boundary-neighbouring cells, blue color means frozen or known value.)

\[
\begin{align*}
    a^u_o (u_o - u^*_o) + \frac{p'_e - p'_o}{\Delta x} &= 0 \\
    a^v_o (v_o - v^*_o) + \frac{p'_n - p'_o}{\Delta y} &= 0 \\
    u_o - u_w &= 0 \\
    v_o - v_s &= 0 \\
    v'_o = v_o - v^*_o &= 0 \\
    u'_w &= 0 \\
    p'_o &= 0 \\
    p'_s &= 0 \\
\end{align*}
\]

The pressure correction eq. for cell next to boundary

\[
\begin{align*}
    u_o &= u^*_o + \frac{p'_e - p'_o}{-a^u_o (u_o) \Delta x} \\
    v_o &= v^*_o + \frac{p'_n - p'_o}{-a^v_o (v_o) \Delta y} \\
    u_w &= u^*_w + \frac{p'_o - p'_w}{-a^u_o (u_w) \Delta x} \\
    v_s &= v^*_s + \frac{p'_o - p'_s}{-a^v_o (v_s) \Delta y} \\
\end{align*}
\]

\[
\begin{align*}
    u_o - u_w &= \frac{1}{\Delta x} \left( u^*_o - u^*_w \right) \\
    v_o - v_s &= \frac{1}{\Delta y} \left( v^*_o - v^*_s \right) \\
\end{align*}
\]

\[
\begin{align*}
    b p'^* &= \frac{u_o - u_w}{\Delta x} + \frac{v_o - v_s}{\Delta y} \\
    \frac{p'_e - p'_o}{-a^u_o (u_o) \Delta x^2} - \frac{p'_o - p'_w}{-a^u_o (u_w) \Delta x^2} + \frac{p'_n - p'_o}{-a^v_o (v_o) \Delta y^2} - \frac{p'_o - p'_s}{-a^v_o (v_s) \Delta y^2} &= -b p'^* 
\end{align*}
\]
The SIMPLE algorithm (continued)
- General expression of the pressure correction equation
  (blue color means frozen or known value.)

\[
\frac{p_e' - p_o'}{-a_o^u(u_o)\Delta x^2} - \frac{p_o' - p_w'}{-a_o^u(u_w)\Delta x^2} + \frac{p_n' - p_o'}{-a_o^v(v_o)\Delta y^2} - \frac{p_o' - p_s'}{-a_o^v(v_s)\Delta y^2} = -b_p'^* \\

a_o^p p_o' + \sum_{nb} a_{nb}^p p_{nb}' + b_p' = 0
\]

(1) The coefficient \(a_o^p\) and \(a_{nb}^p\) depends on the center coefficient, \(a_o^v\) and \(a_o^u\), or the diagonal part of the matrix \(A_{uv}\)

(2) Using full velocity \(u,v\) boundary condition, the source term only depends on \(u^*\) and \(v^*\)

\[
b_p' = \frac{u_o^* - u_w^*}{\Delta x} + \frac{v_o^* - v_s^*}{\Delta y}
\]
Rewrite the derivation of $p'$-eq. in matrix form
blue color means frozen or known value.

Starting from

\[
\begin{align*}
A_{Np \times Nuv} \begin{bmatrix} u \\ v \end{bmatrix} &= -B_{Np} \\
A_{Nuv \times Nuv}^{diag} \left( \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} u^* \\ v^* \end{bmatrix} \right) + A_{Nuv \times Np}[p'] &= 0
\end{align*}
\]

\[
\begin{align*}
\begin{bmatrix} u \\ v \end{bmatrix} &= \begin{bmatrix} u^* \\ v^* \end{bmatrix} - INV(A_{Nuv \times Nuv}^{diag})A_{Nuv \times Np}[p'] \\
A_{Np \times Nuv} \begin{bmatrix} u \\ v \end{bmatrix} &= -B_{Np}
\end{align*}
\]

\[
A_{Np \times Nuv} \left( \begin{bmatrix} u^* \\ v^* \end{bmatrix} - INV(A_{Nuv \times Nuv}^{diag})A_{Nuv \times Np}[p'] \right) = -B_{Np}
\]

The $p'$-eq. in matrix form for interior cells

\[
\left\{ A_{Np \times Nuv} \right\} \left[ p' \right] = A_{Np \times Nuv} \left[ \begin{bmatrix} u^* \\ v^* \end{bmatrix} \right] - B_{Np}
\]

\[
\left\{ A_{Np \times Np}^{p'} \right\} \left[ p' \right] = B_{Np}^{p'}
\]
Viewing pressure-correction eq. with A and B

\[
\left\{ A_{Np \times Nuv} \text{INV}(A_{Nuv \times Nuv}^{diag}) A_{Nuv \times Np} \right\} [p'] = A_{Np \times Nuv} \left[u^*_v\right] - B_{Np}
\]

\[ A_{Nuv \times Np} + A_{Nuv \times Nuv}^{nb} \]
A comment

For incompressible N-S eq, If we specify $u$ and $v$ value at all boundaries, there is no need to specify $p$ value at boundary.

$$\left\{ A_{Np \times Np}^{p'} \right\} [p'] = B_{Np}^{p'}$$

Alternatively, you can also set $p$-value at this p-cell, then the $u$-velocity at this u-cell will become a new unknown ($N_u^{\text{new}} = N_u + 1$). However this p-cell will be exclude from adding a discretized continuity eq.
The SIMPLE algorithm (continued)
- Applying the solved pressure correction $p'$ to update the final $u, v$ and $p$ fields at the end of one iteration step

$$u_o = u_o^* + \frac{p'_e - p'_o}{-a_o^u(u_o)\Delta x}$$
$$v_o = v_o^* + \frac{p'_n - p'_o}{-a_o^v(v_o)\Delta y}$$
$$p_o = p_o^* + p'_o$$

Or, under-relaxation factor for $u, v, p$ can also be applied for convergence

$$u_o = u_o^* + \text{urf}_u \times \frac{p'_e - p'_o}{-a_o^u(u_o)\Delta x}$$
$$v_o = v_o^* + \text{urf}_v \times \frac{p'_n - p'_o}{-a_o^v(v_o)\Delta y}$$
$$p_o = p_o^* + \text{urf}_p \times p'_o$$
Numerical demonstration of SIMPLE algorithm in a 3X3 cells domain

Same B.C. as in yesterday’s lecture, however in this case the interior u,v,p values are randomly initialized. We can easily see the fact continuity eq. is not satisfied at most of p-cells, i.e. the net-mass flow in/out of all 4 cell-surfaces of one p-cell is not zero.
The SIMPLE algorithm at **interior** cells (allowing $b_o^p = 0$)

(blue color means frozen or known value)

Our goal

\[
\begin{align*}
a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb} + \frac{p_e - p_o}{\Delta x} + b_o^u &= 0 \\
a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb} + \frac{p_n - p_o}{\Delta y} + b_o^v &= 0 \\
u_o - u_w + \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

Guessed, known value

approx.

\[
\begin{align*}
a_o^u u_o^* + \sum_{nb} a_{nb}^u u_{nb}^* + \frac{p_e^* - p_o^*}{\Delta x} + b_o^u &= 0 \\
a_o^v v_o^* + \sum_{nb} a_{nb}^v v_{nb}^* + \frac{p_n^* - p_o^*}{\Delta y} + b_o^v &= 0
\end{align*}
\]

Take the difference between ideal eq. and the approx. one

\[
\begin{align*}
a_o^u (u_o - u_o^*) + \frac{p_e' - p_o'}{\Delta x} &= 0 \\
a_o^v (v_o - v_o^*) + \frac{p_n' - p_o'}{\Delta y} &= 0 \\
u_o - u_w + \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

From here, we are going to derive an equation for pressure correction!

\[
\begin{align*}
a_o^u u_o' + \sum_{nb} a_{nb}^u u_{nb}' + \frac{p_e' - p_o'}{\Delta x} &= 0 \\
a_o^v v_o' + \sum_{nb} a_{nb}^v v_{nb}' + \frac{p_n' - p_o'}{\Delta y} &= 0
\end{align*}
\]

$u', v', p'$ are still coupled, difficult to handle, then **neglect neighboring terms** (main assumption)
(SIMPLE: 1-0) Preparation $A_{(N_u+N_v) \times (N_u+N_v)}$ and $B_{(N_u+N_v)}$

$A_{12 \times 12} = 0$

$B_{12} = 0$
(SIMPLE: 1-1) : Assemble $A_{uv}$ and $B_{uv}$ from the 6 u-eq.s without the pressure terms

$$A_{12 \times 12}$$

$$a^u_o u_o + \sum_{nb} a^u_{nb} u_{nb} + \frac{p^* - p^*_o}{\Delta x} + b^u_o = 0 \quad \text{(Loop for all unknown 6 u-cells)}$$

$$a^v_o v_o + \sum_{nb} a^v_{nb} v_{nb} + \frac{p^*_n - p^*_o}{\Delta y} + b^v_o = 0 \quad \text{(Loop for all unknown 6 v-cells)}$$
(SIMPLE: 1-2) : Assemble $A_{uv}$ and $B_{uv}$ from the 6 u-eq.s with the remaining pressure-terms, however using the frozen ($p^*$) pressure values.

$$A_{12 \times 12}$$

$$B_{12}$$

$$a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb} + \frac{p^*_e - p^*_o}{\Delta x} + b_o^u = 0 \quad \text{(Loop for all unknown 6 u-cells)}$$

$$a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb} + \frac{p^*_n - p^*_o}{\Delta y} + b_o^v = 0 \quad \text{(Loop for all unknown 6 v-cells)}$$
(SIMPLE: 1-3) : Assemble $A_{uv}$ and $B_{uv}$ from the 6 v-eq.s without pressure-terms

$$a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb} + \frac{p_e^* - p_o^*}{\Delta x} + b_o^u = 0$$  (Loop for all unknown 6 u-cells)

$$a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb} + \frac{p_n^* - p_o^*}{\Delta y} + b_o^v = 0$$  (Loop for all unknown 6 v-cells)
(SIMPLE: 1-4) : Assemble $A_{uv}$ and $B_{uv}$ from the 6 v-eq.s with the remaining pressure-terms, however using the frozen pressure ($p^*$) values

$$A_{12 \times 12}$$

$$B_{12}$$

$$a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb} + \frac{p_e^* - p_o^*}{\Delta x} + b_o^u = 0 \quad \text{(Loop for all unknown 6 u-cells)}$$

$$a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb} + \frac{p_n^* - p_o^*}{\Delta y} + b_o^v = 0 \quad \text{(Loop for all unknown 6 v-cells)}$$
(SIMPLE: 1-5) Solve $X_{uv}$ to update $u^*$ and $v^*$

Solve $AX + B = 0$ and $X \Rightarrow [u^*; v^*]$

After solve $X_{uv}$, plot $u^*$ and $v^*$ (notice the changes compared with the initial field, also easily notice the net-mass flow in/out of any cell is still not zero)
(SIMPLE: 2-1) : Extract the diagonal part of $A_{uv}$ for assemble the coefficient matrix $A_p$ for the pressure correction equations, i.e. $a_o^u$ and $a_o^v$
(SIMPLE: 2-2) : Prepare and assemble new $A_{p'}$ and $B_{p'}$ of size 9X9 (or 9), to be solved for $X_{p'}$, for the pressure correction ($p'$).

\[
A_{p'}X_{p'} + B_{p'} = 0
\]

\[
b_{p'} = \frac{u_o^* - u_w^*}{\Delta x} - \frac{v_o^* - v_s^*}{\Delta y}
\]

\[
a_o^p p'_o + \sum_{nb} a_{nb}^p p'_{nb} + b_{p'} = 0 \quad \text{Loop for 9 p-cells}
\]
(SIMPLE: 3) : Solve $A_{p'}X_{p'} + B_{p'} = 0$ for $X_{p'}$ of the 9 pressure correction ($p'$) then update $u, v, p$. This complete one interaction.

$u_o = u_o^* + ur f_u \times \frac{p_e' - p_o'}{-a_o^u(u_o) \Delta x}$

$v_o = v_o^* + ur f_v \times \frac{p_n' - p_o'}{-a_o^v(v_o) \Delta y}$

$p_o = p_o^* + ur f_p \times p_o'$

In this example, under-relaxation factors are set to 1. Now, we can easily verify the total in/out mass fluxes over all 4 faces of any p-cell is zero.
(SIMPLE: Outer-iteration) Repeat 1-3 for more iterations

It diverges, if \( urf = 1 \)

\[
\begin{align*}
urfp &= 0.2 \\
urfv &= 0.2 \\
urfu &= 0.2 \\
Urfp &= 0.5 \\
Urfv &= 0.7 \\
Urfu &= 0.7
\end{align*}
\]

Converged, same result as direct/coupled solver
Outline for part 2

• SIMPER-C
• SIMPLE-R
• PISO
The SIMPLE-Consistent algorithm

Van Doormal and Raithby (1984), only modify the u, v correction step.

Perform standard SIMPLE until \( p' \) is ready!

\[
\begin{align*}
a^u_o u'_o + \sum_{nb} a^u_{nb} u'_{nb} + \frac{p'_e - p'_o}{\Delta x} &= 0 \\
a^v_o v'_o + \sum_{nb} a^v_{nb} v'_{nb} + \frac{p'_n - p'_o}{\Delta y} &= 0
\end{align*}
\]

\[
A^\text{diag}_{Nuv \times Nuv} \left( \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} u^* \\ v^* \end{bmatrix} \right) + A_{Nuv \times Np} [p'] = 0
\]

\[
\begin{align*}
&\quad + A^{nb}_{Nuv \times Nuv}
\end{align*}
\]

\[
A_{Nuv \times Nuv} \left( \begin{bmatrix} u \\ v \end{bmatrix} - \begin{bmatrix} u^* \\ v^* \end{bmatrix} \right) + A_{Nuv \times Np} [p'] = 0
\]

Note is another eq. in the style of \( AX = B \) requiring matrix inversion.
Outline for part 2

• SIMPLE-C
• SIMPLE-R
• PISO
The SIMPLER(Revisit) algorithm

**Motivation**: provide a better(?) guessed pressure $p$ to start the standard SIMPLE algorithm.

\[
\begin{align*}
a^u_o u_o^* + \sum_{nb} a^u_{nb} u^*_{nb} + \frac{p^*_e - p^*_o}{\Delta x} + b^u_o &= 0 \\
a^v_o v_o^* + \sum_{nb} a^v_{nb} v^*_{nb} + \frac{p^*_n - p^*_o}{\Delta y} + b^v_o &= 0
\end{align*}
\]

**Solution**: An additional discretized equation for pressure (instead of pressure correction) is solved to directly calculate the pressure, following this step, perform a standard SIMPLE (note: in which contain another pressure-correction equation).
The SIMPLE-R algorithm (the pressure equation)

(blue color means frozen or known value.)

Guessed off-diagonal velocity $u_{nb}^*, v_{nb}^*$

\[
\begin{align*}
    a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb}^* + \frac{p_e - p_o}{\Delta x} + b_o^u &= 0 \\
    a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb}^* + \frac{p_n - p_o}{\Delta y} + b_o^v &= 0 \\
    u_o - u_w \frac{\Delta x}{\Delta x} + v_o - v_s \frac{\Delta y}{\Delta y} &= 0
\end{align*}
\]

\[
\begin{align*}
    \hat{u}_o &= -\left(\sum_{nb} a_{nb}^u u_{nb}^* + b_o^u\right)/a_o^u \\
    \hat{v}_o &= -\left(\sum_{nb} a_{nb}^v v_{nb}^* + b_o^v\right)/a_o^v
\end{align*}
\]

The pressure eq. in SIMPLE-R. algorithm.

It is similarly derived as the pressure-correction eq. in SIMPLE method, just replace $(u^*, v^*)$ by $(\hat{u}, \hat{v})$
The SIMPLE-R algorithm (continued)
(blue color means frozen or known value.)

After solving \( p \) to serve as a better guess. Following after that, append a standard SIMPLE procedure.

Solve the implicit momentum equations for new unknown \( u^* \) and \( v^* \), note \( a, b \) have been already computed at last step.

\[
\begin{align*}
a_o^u u_o^* + \sum_{nb} a_{nb}^u u_{nb}^* + \frac{p_e - p_o}{\Delta x} + b_o^u &= 0 \\
a_o^v v_o^* + \sum_{nb} a_{nb}^v v_{nb}^* + \frac{p_n - p_o}{\Delta y} + b_o^v &= 0
\end{align*}
\]

\[
\begin{align*}
a_o^p p_o^* + \sum_{nb} a_{nb}^p p_{nb}^* + b^p &= 0
\end{align*}
\]

Since \( p \) already satisfy continuity, there is no need to correct it.

\[
\begin{align*}
  u_o &= u_o^* + \frac{p_e^' - p_o^'}{-a_o^u(u_o^*)\Delta x} \\
v_o &= v_o^* + \frac{p_n^' - p_o^'}{-a_o^v(v_o^*)\Delta y} \\
p_o &= p_o - p_o^* + p_o^'
\end{align*}
\]
Demonstration of SIMPLER algorithm in a 3X3 cells domain (SIMPLER: 1-0) : Starting from a initial random guess
(SIMPLER: 1-1) : Assemble $A_{uv}$ and $B_{uv}$ from the 6 u-eq.s and 6 v-e.q.s without the pressure terms

$$A_{12 \times 12}$$

$$B_{12}$$

\[ a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb} + \frac{p_e - p_o}{\Delta x} + b_o^u = 0 \quad \text{(Loop for all unknown 6 u-cells)} \]

\[ a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb} + \frac{p_n - p_o}{\Delta y} + b_o^v = 0 \quad \text{(Loop for all unknown 6 v-cells)} \]
(SIMPLER: 1-2) : Split $A_{uv}$ into a diagonal part $A_{diag}$ (for $a^u_o$ and $a^v_o$ ) and the off-diagonal or neighbouring part $A_{nb}$ (for $a^u_{nb}$ and $a^v_{nb}$ )
The SIMPLE-R algorithm (the pressure equation)
(blue color means frozen or known value.)

Guessed off-diagonal velocity $u_{nb}^*, v_{nb}^*$

\[
\begin{align*}
\frac{a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb}^* + \frac{p_e - p_o}{\Delta x} + b_o^u}{\Delta x} &= 0 \\
\frac{a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb}^* + \frac{p_n - p_o}{\Delta y} + b_o^v}{\Delta y} &= 0
\end{align*}
\]

\[
\begin{align*}
u_o - u_w + \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

The pressure eq. in SIMPLE-R. algorithm. It is similarly derived as the pressure-correction eq. in SIMPLE method, just replace $(u^*, v^*)$ by $(\hat{u}, \hat{v})$
(SIMPLER: 1-3) Using the $A^{diag}$ and $A^{nb}$ to compute $[\hat{u}; \hat{v}]$

$$\hat{u}_o = - \left( \sum_{nb} a_{nb}^u u_{nb}^* + b_o^u \right) / a_o^u$$

$$\hat{v}_o = - \left( \sum_{nb} a_{nb}^v v_{nb}^* + b_o^v \right) / a_o^v$$

$$\Rightarrow [\hat{u}; \hat{v}]^T = - \frac{1}{A^{diag}} (A^{nb} [u^*; v^*]^T + B)$$

Initial random $[u; v]$

Computed $[\hat{u}; \hat{v}]$, notice the vector adjustment due to viscosity and convection term, of course the discretized cont.-eq.s are not satisfied.
(SIMPLER: 1-4) Using $[\hat{u}; \hat{v}]$ and the coefficient stored in $A_{uv}^{diag}$ (size 12x12) we can get the $A_p$ and $B_p$ for 9 pressure equations.

$$A_{p_{9\times9}}$$

$$B_{p_9}$$

$$a_o^p p_o + \sum_{nb} a_{nb}^p p_{nb} + b^p = 0$$
(SIMPLER: 1-5) Solve $A_p X_p + B_p = 0$ to get the new pressure $p$.

[$\hat{u}; \hat{v}$] and the initial random $p$

[$\hat{u}; \hat{v}$] and the new $p$. Note, since $u,v$ have not been updated, the continuity-eq. are still not satisfied.
(SIMPLER: 2-1) : Further update $B_{uv}$ from the pressure terms 6 u-eq.s and 6 v-e.q.s using the recently computed $p$

\[ A_{12 \times 12} \]

\[ B_{12} \]

\[ a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb} + \frac{p_e - p_o}{\Delta x} + b_o^u = 0 \]  
(Loop for all unknown 6 u-cells)

\[ a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb} + \frac{p_n - p_o}{\Delta y} + b_o^v = 0 \]  
(Loop for all unknown 6 v-cells)
(SIMPLER: 2-2) : Solve $A_{uv} X_{uv} + B_{uv} = 0$ to get the intermediate $u^*$ and $v^*$

$X_{uv} \Rightarrow [u^*, v^*]^T$

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$\hat{u} = [0; 0.52; -0.15; 0.14]$ and the new $p$

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$\hat{v} = [0; 0.087; -0.15; 0.078]$ and the new $p$
(SIMPLER: 3) Using \([u^*; v^*]\) and the coefficient stored in \(A_{uv}^{diag}\) (size 12x12) compute \(A_{p'}\) and \(B_{p'}\) to build 9 pressure correction equations.

\[
a^p_{o} p'_o + \sum_{nb} a^p_{nb} p'_{nb} + b^p = 0
\]
(SIMPLER: 4) : Solve $A_{p'} X_{p'} + B_{p'} = 0$ to get the pressure correction $p'$, i.e. $X_{p'} \Rightarrow [p']$, then apply pressure correction to get the new $u, v$.

$$u_o = u_o^* + \frac{p_e - p_o'}{-a_o^u (u_o) \Delta x}$$

$$v_o = v_o^* + \frac{p_n' - p_o'}{-a_o^v (v_o) \Delta y}$$

New $[u, v]$ and the new $p$

Old $[u^*, v^*]$ and the new $p$
(SIMPLER: outer-iteration) Repeat 1-4 for more iterations

All converged, same result as direct/coupled solver
Outline for part 2

• SIMPLE-C
• SIMPLE-R
• PISO
The PISO algorithm (Pressure Implicit with splitting of operator)

Developed originally for non-iterative computation of unsteady compressible flow. It was adapted successfully for iterative solution of steady state problem. It involve one predictor step and two corrector steps, which can be also viewed as a SIMPLE + a further corrector step.

Let’s first check the **main assumption** of SIMPLE:

\[
\begin{align*}
    a_o^u u'_o + \sum_{nb} a_{nb}^u u'_{nb} + \frac{p'_e - p'_o}{\Delta x} &= 0 \\
    a_o^v v'_o + \sum_{nb} a_{nb}^v v'_{nb} + \frac{p'_n - p'_o}{\Delta y} &= 0
\end{align*}
\]

**Ideal:** do not neglect the neighboring coefficients (off-diagonal) terms completely, To achieve this, add one extra corrector step.
The SIMPLE algorithm at interior cells (allowing \( b_o^p = 0 \))
(blue color means frozen or known value.)

Our goal

\[
\begin{align*}
 a_o^u u_o + \sum_{nb} a_{nb}^u u_{nb} + \frac{p_e - p_o}{\Delta x} + b_o^u &= 0 \\
 a_o^v v_o + \sum_{nb} a_{nb}^v v_{nb} + \frac{p_n - p_o}{\Delta y} + b_o^v &= 0 \\
 \frac{u_o - u_w}{\Delta x} + \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

From here, we are going to derive an equation for pressure correction!

Take the difference between ideal eq. and the approx. one

\[
\begin{align*}
 a_o^u (u_o - u_o^*) + \frac{p_e' - p_o'}{\Delta x} &= 0 \\
 a_o^v (v_o - v_o^*) + \frac{p_n' - p_o'}{\Delta y} &= 0 \\
 \frac{u_o - u_w}{\Delta x} + \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

\( u = u^* + u' \)
\( v = v^* + v' \)
\( p = p^* + p' \)

\( u^* + u' = u^{**} \)
\( v^* + v' = v^{**} \)
\( p^* + p' = p^{**} \)

\( u', v', p' \) are still coupled, difficult to handle, then neglect neighboring terms (main assumption)
PISO algorithm

The predictor + 1st corrector step is a standard SIMPLE

( blue color denotes known value, black is unknown )

The ideal eq.s. (i.e. the couple method)

\[
\begin{align*}
    a^u_o u_o + \sum_{nb} a^u_{nb} u_{nb} + \frac{p_e - p_o}{\Delta x} + b^u_o &= 0 \\
    a^v_o v_o + \sum_{nb} a^v_{nb} v_{nb} + \frac{p_n - p_o}{\Delta y} + b^v_o &= 0 \\
    \frac{u_o - u_w}{\Delta x} + \frac{v_o - v_s}{\Delta y} &= 0
\end{align*}
\]

\[
\begin{align*}
    a^u_o u^*_o + \sum_{nb} a^u_{nb} u^*_{nb} + \frac{p_e^* - p^*_o}{\Delta x} + b^u_o &= 0 \\
    a^v_o v^*_o + \sum_{nb} a^v_{nb} v^*_{nb} + \frac{p_n^* - p^*_o}{\Delta y} + b^v_o &= 0 \\
    u^*_o - u^*_w + \frac{v^*_o - v^*_s}{\Delta y} &= 0
\end{align*}
\]

(u^*_o, v^*_o and p^*_o) is the solution from a standard SIMPLE algorithm

Started from guessed \( p^* \) and B.D

\[
\begin{align*}
    a^u_o u_o + \sum_{nb} a^u_{nb} u_{nb} + \frac{p_e - p_o}{\Delta x} + b^u_o &= 0 \\
    a^v_o v_o + \sum_{nb} a^v_{nb} v_{nb} + \frac{p_n - p_o}{\Delta y} + b^v_o &= 0 \\
    u^*_o - u^*_w + \frac{v^*_o - v^*_s}{\Delta y} &= 0
\end{align*}
\]

+ \( u^*_{nb} (v^*_{nb}) \) are known from (different than ideal eq.s)
PISO algorithm viewed as a sequence of operator (blue color denotes known value)

\[
\begin{align*}
    a^u_0 u^*_o + \sum_{nb} a^u_{nb} u^*_{nb} + \frac{p^*_e - p^*_o}{\Delta x} + b^u_o &= 0 \\
    a^v_0 v^*_o + \sum_{nb} a^v_{nb} v^*_{nb} + \frac{p^*_n - p^*_o}{\Delta y} + b^v_o &= 0
\end{align*}
\]

Feed in known \( u^*_{nb} (v^*_{nb}) \)

Standard SIMPLE return \((u^{**}, v^{**}, p^{**})\) here

\[
\begin{align*}
    a^u_0 u^{**}_o + \sum_{nb} a^u_{nb} u^{**}_{nb} + \frac{p^{**}_e - p^{**}_o}{\Delta x} + b^u_o &= 0 \\
    a^v_0 v^{**}_o + \sum_{nb} a^v_{nb} v^{**}_{nb} + \frac{p^{**}_n - p^{**}_o}{\Delta y} + b^v_o &= 0 \\
    \frac{u^{**} - u^*_w}{\Delta x} + \frac{v^{**} - v^*_s}{\Delta y} &= 0
\end{align*}
\]

Feed in known \( u^{**}_{nb} (v^{**}_{nb}) \)

PISO add an extra level of such eq. series to return \((u^{***}, v^{***}, p^{***})\)

\[
\begin{align*}
    a^u_0 u^{***}_o + \sum_{nb} a^u_{nb} u^{***}_{nb} + \frac{p^{***}_e - p^{***}_o}{\Delta x} + b^u_o &= 0 \\
    a^v_0 v^{***}_o + \sum_{nb} a^v_{nb} v^{***}_{nb} + \frac{p^{***}_n - p^{***}_o}{\Delta y} + b^v_o &= 0 \\
    \frac{u^{***} - u^{**}_w}{\Delta x} + \frac{v^{***} - v^{**}_s}{\Delta y} &= 0
\end{align*}
\]

Feed in known \( u^{***}_{nb} (v^{***}_{nb}) \)

It is obvious the more series of such system of equation becomes (with more stars), the more it approaches the ideal direct-coupled eq.s)
The difference between two successive sequence is (define \( p'' = p^{**} - p^{*} \))

\[
\begin{align*}
a_o^u (u^{***} - u^{**}) + \sum_{nb} a_{nb}^u (u_{nb}^{**} - u_{nb}^{*}) + \frac{p_e'' - p_o''}{\Delta x} &= 0 \\
a_o^v (v^{***} - v^{**}) + \sum_{nb} a_{nb}^v (v_{nb}^{**} - v_{nb}^{*}) + \frac{p_n'' - p_o''}{\Delta y} &= 0 \\
\frac{u^{***} - u^{**}}{\Delta x} + \frac{v^{***} - v_s^{**}}{\Delta y} &= 0
\end{align*}
\]

Define a corrected velocity

\[
\begin{align*}
u_o^# &= u_o^{**} - \sum_{nb} a_{nb}^u (u_{nb}^{**} - u_{nb}^{*})/a_o^u \\
v_o^# &= v_o^{**} - \sum_{nb} a_{nb}^v (v_{nb}^{**} - v_{nb}^{*})/a_o^v
\end{align*}
\]

Similarly derive a 2nd pressure-correction eq. from \( p'' \) as shown in the \( p' \) equation in standard SIMPLE, as long as \((p'', u^#, u^{***})\) are replaced by \((p', u^*, u^{**})\).
Summarized iteration procedure of PISO algorithm

(1) Predictor + 1\textsuperscript{st} corrector: apply standard SIMPLE,
    (1.a) starting from guessed $p^*$,
    (1.b) get $u^*$ and $v^*$
    (1.3) solve pressure correction eq. ($p'$) to correct $u^{**}$, $v^{**}$, $p^{**}$

(2) 2\textsuperscript{nd} corrector
    (2.1) compute another intermediate velocity field according to
    
    $u_0^\# = u_0^{**} - \sum_{nb} a_{nb}^{u} (u_{nb}^{**} - u_{nb}^*) / a_0^{u}$
    
    $v_0^\# = v_0^{**} - \sum_{nb} a_{nb}^{v} (v_{nb}^{**} - v_{nb}^*) / a_0^{v}$
    
    (2.2) Build 2\textsuperscript{nd} pressure-correction $p''$ eq. according to, solve $p''$ and then correct the $p^{***}$, $u^{***}$ and $v^{***}$

**Note:** $a_0^{u}$ and $a_0^{v}$ are the same as in the 1\textsuperscript{st} $p'$-eq.
Demonstration of PISO algorithm in a 3X3 cells domain

**PISO (predictor+1\textsuperscript{st} corrector)**: Starting from an initial random guessed p\textsuperscript{*} field get \(u^{**}, v^{**}, p^{**}\)

---

**Initial** random guessed p\textsuperscript{*} field, [initial guessed u,v have no use in SIMPLE, except at B.D. ]

Now, rename this field by two star (\(u^{**}, v^{**}, p^{**}\))
(PISO : predictor+1st corrector)

The following *repeats* the standard SIMPLE procedure

1) Starting with guessed $p^*$, Assemble $A_{uv}$ and $B_{uv}$ from the 6 u-eq.s and 6 v-e.q.s (including $p$ term), Solve $A_{uv}X_{uv} + B_{uv} = 0$

to get $X_{uv} \Rightarrow [u^*; v^*]$

\[ a_o u_o^* + \sum_{nb} a_{nb} u_{nb}^* + \frac{p_e - p_o^*}{\Delta x} + b_o^u = 0 \]  
(Loop for all unknown 6 u-cells)

\[ a_o v_o^* + \sum_{nb} a_{nb} v_{nb}^* + \frac{p_n - p_o^*}{\Delta y} + b_o^v = 0 \]  
(Loop for all unknown 6 v-cells)

Note: this $[u^*; v^*]$ and the split matrix $A_{uv} = A_{uv}^{\text{diag}} + A_{uv}^{\text{nb}}$ will be also used in the PISO: 2nd corrector step.

2) Assemble and solve a pressure-correction equation for the 9 $p -$ cells, solve $A_{p'}X_{p'} + B_{p'} = 0$ to get $X_{p'} \Rightarrow [p']$ of 9 pressure c

3) Using $p'$ to update $u, v, p$. (Here complete standard SIMPLE) For continue PISO step, rename them as $u^{**}, v^{**}, p^{**}$.

\[
\begin{align*}
  u_o^{**} &= u_o^* + u rf_u \times \frac{p_e - p_o'}{-a_o^u(u_o)\Delta x} \\
  v_o^{**} &= v_o^* + u rf_v \times \frac{p_n' - p_o'}{-a_o^v(v_o)\Delta y} \\
  p_o^{**} &= p_o^* + u rf_p \times p_o'
\end{align*}
\]
(PISO: 2nd corrector- 1) : Compute the difference between \( u^{**} \) and \( u^* \) (also for \( v \)), i.e. \( \begin{bmatrix} u^{**} - u^* \\ v^{**} - v^* \end{bmatrix} \), then compute

\[
\begin{bmatrix} u^* \\ v^* \end{bmatrix} = \begin{bmatrix} u^{**} \\ v^{**} \end{bmatrix} - \frac{1}{A_{uv}^{diag}} A_{nb}^{uv} \begin{bmatrix} u^{**} - u^* \\ v^{**} - v^* \end{bmatrix}
\]

\[
\begin{align*}
u^*_o &= \nu^{**}_o - \sum_{nb} a_{nb}^u (\nu^{**}_{nb} - \nu^*_{nb})/a_{o}^{\nu} \\
u^*_o &= \nu^{**}_o - \sum_{nb} a_{nb}^v (\nu^{**}_{nb} - \nu^*_{nb})/a_{o}^{\nu}
\end{align*}
\]

From the end of PISO 1st corrector, \( \begin{bmatrix} u^{**} \\ v^{**} \\ p^{**} \end{bmatrix} \) from the end of PISO 1st corrector is a standard SIMPLE, it must satisfy continuity eq.

Note \( \begin{bmatrix} u^* \\ v^* \\ p^* \end{bmatrix} \) from 2nd PISO-corrector step are just the an intermediate velocity, it does not satisfy continuity eq.
(PISO: 2\textsuperscript{nd} Corrector – 2) : Build $A_{p''}$ and $B_{p''}$ of size 9X9 (or 9), to be solved for $X_{p''}$ for the 2\textsuperscript{nd} pressure correction ($p''$).

$$A_{p''}X_{p''} + B_{p''} = 0$$

Same as 1\textsuperscript{st} $p' - eq.$

$$a_{o}^{p'} p_{o}^{''} + \sum_{nb} a_{nb}^{p'} p_{nb}^{''} + b^{p''} = 0$$

Loop for 9 p-cells
(PISO: 2nd corrector - 3) : Applying 2nd pressure correction \( (p'') \) to correct \([u^#, v^#, p^{**}]\) and calculate the \([u^{***}, v^{***}, p^{***}]\). It then complete one PISO step.

\[
\begin{align*}
    u^{***} &= u^# + urf_u \times \frac{p''_e - p''_o}{-a^u_o(u_o)\Delta x} \\
    v^{***} &= v^# + urf_v \times \frac{p''_n - p''_o}{-a^v_o(v_o)\Delta y} \\
    p^{***} &= p^{**} + urf_p \times p''_o
\end{align*}
\]

Notice the difference is small, but it now satisfy the exact continuity eq.
(PISO: outer-iteration) Repeat PISO for more iterations

$Ur_f^p = 1$
$Ur_f^v = 1$
$Ur_f^u = 1$

$Ur_f^p = 0.2$
$Ur_f^v = 0.2$
$Ur_f^u = 0.2$

$Ur_f^p = 0.5$
$Ur_f^v = 0.7$
$Ur_f^u = 0.7$

All converged, same result as direct/coupled solver
General comments for pressure-velocity de-coupling methods

• In general
  – SIMPLER is 30% expensive than SIMPLE, but converge faster, reduce total cost 30%-50%
  – SIMPLEC/PISO is shown to be efficient as SIMPLER, but not clear where it can be categorically stated that they are better than SIMPLER.
    • Depends on local variation of density variation (solve additional scalar equation), under-relation, numerical implementation details,
  – Jiang (1986) shows momentum equation is not coupled to scalar variable, PISO is robust and more effective than SIMPLER/SIMPLEC.

• In practice (ANSYS fluent)
  – if you run small case and if the “couple” solver is affordable, it suppose provides the best convergence.
  – Be aware, the failing of finding any converged solver may be due to the physical problem setup (for instance, you should not expect a steady-state solution for a high-Reynold flow without turbulence model, such physical flow is intrinsically unstable.)